1. INTRODUCTION

Repetitive control is a very useful strategy for tracking periodic signals and/or eliminating periodic disturbances. However, since a repetitive controller usually has a large time delay, the design of a low-order robust repetitive controller is difficult, and system design has mainly focused on the nominal plant (Hara et al., 1988; Tomizuka et al., 1989). However, there has been significant progress in robust control theory recently, and some interesting results on robust repetitive control have been obtained by Hoshi et al. (1993), Ishibashi et al. (1994), Hara et al. (1994b) and Shaw and Srinvasan (1993).

Motivated by the recent development of the sampled-data $H^\infty$ control theorem by such researchers as Bamieh and Pearson (1992), Fujioka and Hara (1993), Hara et al. (1994a), Hayakawa et al. (1992), and Kabamba and Hara (1993), and of the static output feedback $H^\infty$ control theorem (de Souza and Xie, 1992), we present a design method for digital repetitive control systems which robustly stabilizes an uncertain plant. The present design method features the lowest order of the repetitive feedback controller. Throughout this paper, $\lambda$ denotes a delay operator, and $\mathbb{R}[\lambda]$ denotes a ring of polynomials in $\lambda$.

2. PROBLEM FORMULATION

Let us consider the partial-static-state-feedback repetitive control system shown in Fig. 1, where the repetitive controller $C_R$.

\[
\begin{align*}
C_R &= \begin{bmatrix} f_1 & \cdots & f_L \end{bmatrix} \\
F &= \begin{bmatrix} f_1 & \cdots & f_L \end{bmatrix}
\end{align*}
\]

\[
C = \begin{bmatrix} f_1 & \cdots & f_L \end{bmatrix} \begin{bmatrix} \lambda^L & \cdots & \lambda \end{bmatrix}
\]

(1)

can have a controllable canonical structure

\[
C_R = \begin{bmatrix} 0 & I_{L-1} \end{bmatrix} \begin{bmatrix} 0 \\ F \end{bmatrix}
\]

(2)

or an observable canonical structure

\[
C_R = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} I_{L-1} & 0 \end{bmatrix} \begin{bmatrix} F \\ 0 \end{bmatrix}
\]

(3)

Fig. 1. Configuration of TDF repetitive control system.
Consider the structurally uncertain plant \( P(s) \) (Asai and Hara, 1992)

\[
\begin{align*}
\dot{x}(t) &= (A + \Phi(t)\Psi_1) x(t) + (B + \Phi(t)\Psi_2) u(t) \\
y(t) &= C x(t) \\
v(t) &= H x(t) \\
\dot{\gamma}(t) &\leq I, \quad E : \text{non-singular},
\end{align*}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R} \) is the control input, \( y(t) \in \mathbb{R} \) is the output and \( v(t) \in \mathbb{R}^n \) is the available partial state.

The nominal plant \( P_0 \) corresponding to \( \Gamma(t) = 0 \) is

\[
\begin{align*}
\dot{x}_0(t) &= A_0 x_0(t) + B_0 u(t) \\
y_0(t) &= C_0 x_0(t) \\
v_0(t) &= H_0 x_0(t).
\end{align*}
\]

To make this problem solvable, we need two assumptions.

**Assumption 1:** \( \{E^T A \ E^T B\} \) is stabilizable.

**Assumption 2:** The sampling period \( \tau \) is chosen such that \( \int_0^\tau e^{-\lambda(t-t)}E^T B dt \) is stabilizable.

Here, we represent the feedback controller in Fig. 1 by

\[
\mathcal{G}_1: \begin{cases}
x_i(t) = A_i x_i(t) + B_i u_i(t) \\
y_i(t) = C_i x_i(t) \\
v_i(t) = H_i C_i x_i(t),
\end{cases}
\]

and the computational delay by

\[
\mathcal{G}_2: \begin{cases}
x_i(t) = A_i x_i(t) + B_i u_i(t) \\
u(t) = H_i C_i x_i(t),
\end{cases}
\]

where

\[
\begin{bmatrix}
A_i \\
B_i \\
C_i \\
D_i
\end{bmatrix} = \begin{bmatrix}
0 & L_{-1} & 0 \\
1 & 0 & F \\
0 & 0 & 0 \\
0 & 0 & F
\end{bmatrix}
\]

Then the closed-loop system with zero input is represented by

\[
\begin{align*}
\dot{x}(t) &= (A + \Phi(t)\Psi_1) x(t) + (B + \Phi(t)\Psi_2) y(t) \\
y(t) &= C x(t) \\
v(t) &= H x(t) \\
\dot{\gamma}(t) &\leq I
\end{align*}
\]

The robust stability of this repetitive control system can be defined as follows (Hara et al., 1994b):

\textbf{Definition} The repetitive control system in Fig. 1 is said to be robustly stable if the closed loop system \( \Sigma \) is asymptotically stable.

Now the design problem for repetitive control systems can be stated as:

i) Design a static feedback gain \( \{F_p, F\} \) which robustly stabilizes the system in Fig. 1.

ii) Design a feedforward controller \( \Phi \), which yields the desired transient input-output response.

3. DESIGN OF FEEDBACK CONTROLLER

Redrawing Fig. 1 as Fig. 2 with the input \( r[i] = 0 \), we obtain the condition for robust stability by applying the small gain theorem (Sivashankar and Khargonekar, 1993):

\[ \|G_{sl}\| < 1, \quad (10) \]

where

\[ \|G_{sl}\| := \sup_{s \in \mathbb{j} \mathbb{C}} \|G_{sl}(s)\| = \sup_{w \in \mathbb{L}} \|G_{sl}(w)\|. \quad (11) \]

To guarantee the robust stability of the system and obtain the desired closed-loop performance, we extend the controlled output and include the weighted control input and the states of the plant, controller and computational delay in it. Now the design problem for the feedback controller can be formulated as (see Fig. 3):

\textbf{Find a static feedback gain} \( \{F_p, F\} \) \textbf{which} internally stabilizes \( \mathcal{G}_1 \) \textbf{and satisfies}

\[ \|G_{sl}\| < 1, \quad (12) \]

where \( \mathcal{G}_2 \) is given by

\textbf{Fig. 2. Robust stability.}
Applying Theorem 2 of Fujioka and Hara (1993) to our case, we have the following theorem.

**Theorem 3.1** The following are equivalent:

i) The system of Fig. 3 is internally stable and $\|G_\infty\|_\infty < 1$.

ii) The system of Fig. 4 is internally stable and $\|G_\infty\|_\infty < 1$.

The discrete equivalent plant $\mathcal{P}_e(\lambda)$ in Fig. 4 is obtained by the following procedure. First, decompose the plant $\mathcal{P}_r$ into two subsystems, $\mathcal{P}_r$ and $\mathcal{P}_r(\lambda)$, as shown in Fig. 5. Secondly, lift the sub-system $\mathcal{P}_r$ and reduce it to a finite-dimensional discrete system. Finally, combining this system with $\mathcal{P}_e(\lambda)$ yields the discrete equivalent plant $\mathcal{P}_e(\lambda)$.) A special case of this problem ($\Psi_e = 0$) can be calculated using the MATLAB $\mu$-toolbox (Balas, et al., 1993.) Let the equivalent general discrete-time plant be

$$
\mathcal{P}_e = 
\begin{bmatrix}
    E^{-1}A & E^{-1}B \mp C\hat{A}_d \\
    B_d S \left[ \begin{array}{c}
        0 \\
        H \end{array} \right] & A_d & 0 \\
    0 & 0 & A_d
\end{bmatrix}
\left[ \begin{array}{c}
    E^{-1}\Phi \\
    0 \\
    0
\end{array} \right]
\begin{bmatrix}
    0 \\
    0 \\
    B_d
\end{bmatrix}
$$

(13)

Then the design problem is equivalent to the following static-output-feedback discrete-time $H^\infty$ control problem:

Find a static output feedback gain $[F_r \ F]$ which internally stabilizes $\mathcal{P}_e(\lambda)$ and satisfies (see Fig. 4)

$$
\|G_\infty\|_\infty < 1.
$$

(15)

The resulting $[F_r \ F]$ can be calculated by the following algorithm (de Souza and Xie, 1992):

**Algorithm 3.1** Set

1. Set the iteration index $i = 1$ and $Q = 0$.
2. Solve the following discrete algebraic Riccati equation:

$$
\begin{bmatrix}
\hat{B} & \hat{B}_2 \\
\end{bmatrix}
\begin{bmatrix}
    \hat{D} & D_{11} & D_{12} \\
    \end{bmatrix}
\begin{bmatrix}
    0 \\
    I \ 0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    I \ 0
\end{bmatrix}
\begin{bmatrix}
    0 \\
    0
\end{bmatrix}
$$

(16)

Step 1: Set the iteration index $i = 1$ and $Q = 0$.

Step 2: Solve the following discrete algebraic Riccati equation:

$$
A_r^T P_r A_r - P_r - (A_r^T P_r \hat{B} + C_r^T \hat{D})
\quad + (\hat{B}^T P_r \hat{B} + \hat{D}^T C_r^T D_{11} + Q + C_r^T C_r) = 0
$$

(17)

for $P_r$. If $P_r \geq 0$, then go to step 3; otherwise, no feasible solution was found.

---

Fig. 3. Formulation of feedback control problem.

Fig. 4. Equivalent discrete-time system.
Step 3: If \( \| P \| < \varepsilon \), where \( \varepsilon \) is a small positive real number, proceed to the next step; otherwise, let \( i = i + 1 \) and set
\[
\Pi_{i-1} = (\hat{B}P_{i-1}\hat{B} + \hat{R})^{-1},
\]
\[
\alpha(P_{i-1}) = \Pi_{i-1}(\hat{B} \hat{P}_{i-1}, \hat{D} \hat{C}i) ; \quad \beta(P_{i-1}) = \Pi_{i-1} S T .
\] (18)
\[
L_i = \beta^{-1}(P_{i-1}) \alpha(P_{i-1}) ; \quad Q_i = L_i^T \Sigma_i L_i .
\]
If \( Q_i < 1 \), set \( Q_i = Q_{i-1} \). Go back to step 2.

Step 4: If \( V_i = I - (B_{i-1}^T P_{i-1} B_{i-1} + D_{i-1}^T D_{i-1}) > 0 \), then go step 5; otherwise, no feasible solution was found.

Step 5: Calculate the feedback gain by
\[
[F_P \quad F] = -\alpha(P_i) C_{i-1}^T (C_i C_{i-1}^T)^{-1}.
\] (19)

4. DESIGN OF FEEDFORWARD CONTROLLER

Assume an \( L \)-periodic signal to be
\[
r(\lambda) = \frac{r_0 + r_1 \lambda + \cdots + r_{m-1} \lambda^{m-1}}{1 - \lambda^L}.
\] (20)

Let \( G_0(\lambda) \) denote the pulse transfer function of the nominal plant with a \( \kappa \)-step computational delay, and \( G_1(\lambda) \) denote the pulse transfer function of \( G_0(\lambda) \) with partial state feedback \( F_P \). Also, let their coprime factorizations be
\[
G_0 = \frac{N}{D_0}, \quad \alpha = N \lambda^b b ; \quad D_0 = \alpha.
\] (21)

and
\[
G_1 = \frac{N}{D}, \quad \alpha = N \lambda^b b ; \quad D = \alpha.
\] (22)

For simplicity, we assume that no cancellation between the zeros and poles of the plant is brought about by the introduction of the partial state feedback \( F_P \).

We can design a low-ripple deadbeat feedforward controller, \( K_1 \), in accordance with \( G_0 \) and \( G_1 \). The term “low-ripple deadbeat response” means that the following conditions are satisfied:

(i) **Deadbeat condition:** The tracking error between the reference input and the output is a finite polynomial, i.e.
\[
e = r - y = \sum_{\mu=1}^{\infty} \mu x^\mu ,
\]
where \( \mu \) is a finite positive integer.

(ii) **Low-ripple condition:** The transfer function from the reference input to the control input is a finite polynomial, i.e. \( G_{z1} = R[\lambda] \).

Some of the details are given in She (1993). Here the main results are just summarized in the following three theorems.

**Theorem 4.1** The parameter \( K_1 \) which yields low-ripple, repetitive deadbeat control with a minimum settling time is given by
\[
K_1 = \frac{1 - (1 - \lambda^L)^{N}}{N},
\] (23)
where \( f' \) is the polynomial
\[
f' := f'_0 + f'_1 \lambda + \cdots + f'_{m-1} \lambda^{m-1},
\] (24)
and its coefficients are determined by the following algorithm (For simplicity, we assume that \( b(\lambda) = 0 \) has only simple roots and let \( \xi_0, \xi_1, \ldots, \xi_m \) denote these roots):  

1) \( f'_0, f'_1, \ldots, f'_{m-1} \) are determined by the \( m \)-multiple original zero of \( G_0 \).

If \( L > m - 1 \),
\[
f'_i = \begin{cases} \lambda^L & i = 0 \\ 0 & i = 1, 2, \ldots, m - 1 \end{cases}.
\] (25)

If \( L \leq m - 1 \), let \( \eta \) be a positive integer which satisfies \( \eta \leq \frac{m - 1}{L} \). Then
\[
f'_i = \begin{cases} \lambda^L & i = kL (k = 0, 1, \ldots, \eta) \\ 0 & i \neq kL (k = 0, 1, \ldots, \eta) \quad \& \quad i \leq m - 1 \end{cases}.
\] (26)

2) \( f'_0, f'_{m-1}, \ldots, f'_{m-1} \) are determined by the \( \ell \)-simple zeros, \( \xi_0, \xi_1, \ldots, \xi_m \), of \( G_0 \):
\[
\begin{bmatrix} f'_0 \\ f'_{m-1} \\ \vdots \\ f'_{m-1} \end{bmatrix} = \begin{bmatrix} 1 & \xi_0 & \xi_0^2 & \cdots & \xi_0^{m-1} \\ 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_m & \xi_m^2 & \cdots & \xi_m^{m-1} \end{bmatrix}^{-1} \begin{bmatrix} g(\xi_0) \\ g(\xi_1) \\ \vdots \\ g(\xi_m) \end{bmatrix},
\]
where
\[
g(\lambda) = \frac{1}{\lambda^L (1 - \lambda^L)} \sum_{i=0}^{m-1} f_i \lambda^i.
\] (28)

**Theorem 4.2** All \( K_1 \) that yield low-ripple repetitive deadbeat control can be parameterized as
\[
K_1 = K_1^c + (1 - \lambda^L) \overline{K}_1 ; \quad \overline{K}_1 \in R[\lambda].
\] (29)

where \( \overline{K}_1 \) is any polynomial.

To optimize the transient response, we choose an appropriate non-zero polynomial
\[
K_1 = K_0 + K_1^c + \cdots + K_1^c + \sum_{i=0}^{m} \xi_i \lambda^i \neq 0,
\] (30)
and use it to optimize the following performance:

$$J = \sum_{i=0}^{n_{\text{max}}-q-1} \| \mathbf{e}[i] \| + \rho^2 \| \Delta u[i] \| ,$$

(31)

where

$$\mathbf{e}[i] := r[i] - y[i]$$

and $\rho$ is a weighting coefficient. The resulting $\mathbf{K}_i$ is given by the following theorem.

**Theorem 4.3** The coefficients of $\mathbf{K}_i$,

$$\mathbf{K}_i := [\mathbf{F}_0 \mathbf{F}_1 \ldots \mathbf{F}_{m_{\text{max}}-q-1}] \in \mathbb{R}^{m_{\text{max}}-q} ,$$

(33)

that minimize the transient response performance index (31) are given by

$$\mathbf{K}_i = \mathbf{F}^* F_2 ,$$

(34)

where

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_0 & \mathbf{F}_1 & \ldots & \mathbf{F}_{m_{\text{max}}-q-1} \end{bmatrix} \in \mathbb{R}^{m_{\text{max}}+q} ,$$

(35)

and

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{F}_0 & -\rho L \mathbf{F}_0 & 0 & \ldots & 0 \; \\
\; \mathbf{F}_1 & \mathbf{F}_0 & -\rho L \mathbf{F}_1 & 0 & \ldots & 0 \; \\
\; \vdots & \; \vdots & \; \ldots & \; \vdots & \; \vdots \; \\
\; 0 & \; \ldots & \; 0 & \mathbf{F}_{m_{\text{max}}-q-1} \end{bmatrix} ,$$

$$\mathbf{F}_0 = \begin{bmatrix} \phi_0 & 0 & \ldots & 0 \\ \phi_1 & \phi_0 & 0 & \ldots & 0 \\ \vdots & \; \vdots & \; \ldots & \; \vdots \; \\
0 & \; \ldots & \; 0 & \phi_0 \end{bmatrix} \in \mathbb{R}^{m_{\text{max}}+q} ,$$

(36)

and

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{F}_0 & \mathbf{F}_1 & \ldots & \mathbf{F}_{m_{\text{max}}-q-1} \end{bmatrix} \in \mathbb{R}^{m_{\text{max}}+q} ,$$

(37)

The minimum of $J$ is given by

$$\min J = J_u = F_2^* F_2 ,$$

(40)

where

$$J_u = \| \mathbf{e}[i] \| + \rho^2 \| \Delta u[i] \| = \sum_{i=0}^{n_{\text{max}}-2} \| e[i] \| + \rho^2 \sum_{i=0}^{n_{\text{max}}-2} \| \Delta u[i] \| .$$

(42)

### 5. NUMERICAL EXAMPLE

Consider the second-order plant

$$P(s) = \frac{\omega^2}{s^2 + 2 \zeta \omega s + \omega^2} .$$

(43)

We assume that

$$\omega = 1 \text{ (rad/s)}$$

(44)

$$0 \leq \zeta(t) \leq 1; \quad \zeta_0 = 0.5,$$

all the states are available, and there is no computational delay, i.e.

$$\kappa = 0 .$$

(45)

Now we design a TDF repetitive controller which robustly stabilizes the control system and whose output tracks the periodic input

$$r(t) = \sin \frac{2 \pi}{2.1} t + \sin 4 \frac{4 \pi}{2.1} t ,$$

(46)

without steady-state error at the sampling points. The design is carried out under the following conditions:

$$\tau = 0.1 \text{ (s)}$$

(47)

$$L = 21$$

$$R_{1/2} = 0; \quad Q_{1/2} = C; \quad Q_{1/2} = I_{21 \times 21}$$

We assume that

$$\omega = 1 \text{ (rad/s)}$$

(44)

$$0 \leq \zeta(t) \leq 1; \quad \zeta_0 = 0.5,$$

all the states are available, and there is no computational delay, i.e.

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Now we design a TDF repetitive controller which robustly stabilizes the control system and whose output tracks the periodic input

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(46)

without steady-state error at the sampling points. The design is carried out under the following conditions:

$$\tau = 0.1 \text{ (s)}$$

(47)

$$L = 21$$

$$R_{1/2} = 0; \quad Q_{1/2} = C; \quad Q_{1/2} = I_{21 \times 21}$$

(47)

The simulation results are shown in Figs. 6-9. In Fig. 6, $\zeta$ has the nominal value 0.5. As expected, the output reaches the steady state from the third cycle and tracks the reference input without steady state error. The control input during the transient is also moderately restricted. For comparison, we plot the response of the minimal-settling-time controller $K_i = K_i^*$ with $\zeta = 0.5$ in Fig. 9. It is clear that there were extreme changes in the control input during the transient.

### 6. CONCLUSIONS

This paper describes a design methodology for digital repetitive control systems in the case where there are some structured uncertainties in the plant. A TDF repetitive control system configuration is employed and a method of designing this system is presented. The advantage of the proposed method is that the
feedback controller has the lowest order. The validity of the present method has been demonstrated by simulations.

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REFERENCES


